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Existence–uniqueness and continuation theorems for stochastic functional differential equations ☆

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Abstract

The main aim of this paper is to develop some basic theories of stochastic functional differential equations (SFDEs). Firstly, we establish stochastic versions of the well-known Picard local existence–uniqueness theorem given by Driver and continuation theorems given by Hale and Driver for functional differential equations (FDEs). Then, we extend the global existence–uniqueness theorems of Wintner for ordinary differential equations (ODEs), Driver for FDEs and Taniguchi for stochastic ordinary differential equations (SODEs) to SFDEs. These show clearly the power of our new results.

Keywords: Stochastic functional differential equations; Existence; Uniqueness; Continuation theorem

1. Introduction

Stochastic differential equations (SDEs) play a very important role in formulation and analysis in mechanical, electrical, control engineering and physical sciences, economic and social sciences. Therefore, the theory of SDEs has been developed very quickly. Recently, the investigation for SFDEs has attracted the considerable attention of researchers and many qualitative theories of SFDEs have been obtained. Many important results can be found in [1–19] and references cited therein. To the best of our knowledge, most of the results on existence theory for

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SODEs and SFDEs focused on developing the global existence–uniqueness to avoid the continuation of the solutions. The important representative works are as follows.

Friedman [2] considered the following SODE of Itô-type

$$dx(t) = f\big(t, x(t)\big)\,dt + g\big(t, x(t)\big)\,d\omega(t), \quad t \in [0, a], \tag{1}$$

with the initial condition

$$x_0 = \xi, \tag{2}$$

where $a > 0$ is a constant. Employing the quasi-local Lipschitz condition: for each $k = 1, 2, \ldots,$ there is a constant $c_k > 0$ such that

$$\big|f(t, x) - f(t, \hat{x})\big| \vee \big|g(t, x) - g(t, \hat{x})\big| \leqslant c_k|x - \hat{x}|, \tag{3}$$

for all $t \in [0, a]$ and those $x, \hat{x} \in R^n$ with $|x| \vee |\hat{x}| \leqslant k$, and the linear growth condition

$$\big|f(t, x)\big| \vee \big|g(t, x)\big| \leqslant c\big(1 + |x|\big), \quad \forall t \in [0, a],\ x \in R^n, \tag{4}$$

where c is a constant, Friedman gave the global existence–uniqueness theorem [2, Theorem 2.2, p. 104] (the earlier works can be found in [3,4]).

In [19], Mao generalized the above result in [2] to the following SFDE of Itô-type

$$dx(t) = B(t, x_t)\,dt + \sigma(t, x_t)\,d\omega(t), \quad t_0 \leqslant t < T, \tag{5}$$

with the initial condition

$$x_{t_0} = \xi, \tag{6}$$

where $x_t(s) = x(t + s)$, $s \in [-\tau, 0]$, $\tau > 0$, T is a constant, or $T = \infty$. Mao [19, Theorem 2.5, p. 153] obtained the global existence–uniqueness of solutions of the initial value problem (5) and (6) if $B(t, x_t)$ and $\sigma(t, x_t)$ satisfy the quasi-local Lipschitz condition[1]: for each $k = 1, 2, \ldots,$ there is a constant $c_k > 0$ such that

$$\big|B(t, \varphi) - B(t, \psi)\big| \vee \big|\sigma(t, \varphi) - \sigma(t, \psi)\big| \leqslant c_k\|\varphi - \psi\|, \tag{7}$$

for all $t \in [t_0, T)$ and those $\varphi, \psi \in C([-\tau, 0], R^n)$ with $\|\varphi\| \vee \|\psi\| \leqslant k$, and the linear growth condition

$$\big|B(t, \varphi)\big| \vee \big|\sigma(t, \varphi)\big| \leqslant c\big(1 + \|\varphi\|\big), \quad \forall t \in [t_0, T),\ \varphi, \psi \in C\big([-\tau, 0], R^n\big), \tag{8}$$

where c is a constant.

[1] It is called the local Lipschitz condition in [19]. We call it the quasi-local Lipschitz condition to differentiate from the ordinary local Lipschitz condition.

However, the quasi-local Lipschitz condition and the linear growth condition are somewhat restrictive and many SFDEs do not obey them. Recently, using a generalized Lipschitz condition and a generalized linear growth condition, Taniguchi [8] discussed the existence–uniqueness of solutions of the SODE (1) and (2). Employing the quasi-local Lipschitz condition (7) and a Lyapunov function, Shen, Luo and Mao [10] dealt with the existence–uniqueness of solutions of the SFDE (5) and (6) without the linear growth condition (8). The papers [8] and [10] presented the local existence theorems of solutions of the SODEs and the SFDEs, respectively. However, since there is no the stochastic version of continuation theorem, it is inconvenient to obtain the global existence of solutions by using the results on the local existence. Motivated by the above discussions, our first aim is to establish stochastic versions of Picard local existence–uniqueness theorem and continuation theorems for SFDEs.

On the other hand, as is well known, the following Wintner theorem [20,26] for ODEs is fundamental one to assure the global existence of solutions of ODEs

$$\dot{u} = U(t, u), \qquad u(t_0) = u_0. \tag{9}$$

Wintner theorem. *(See [20, p. 29].) Let $U(t, u)$ be continuous for $t_0 \leqslant t \leqslant t_0 + a$, $u \geqslant 0$, and let the maximal solution of* (9)*, where $u_0 \geqslant 0$, exist on $[t_0, t_0 + a]$. Let $f(t, y)$ be continuous on the strip $t_0 \leqslant t \leqslant t_0 + a$, y arbitrary, and satisfy*

$$\left|f(t, y)\right| \leqslant U\big(t, |y|\big).$$

Then the maximal interval of existence of solutions of

$$\dot{y} = f(t, y), \qquad y(t_0) = y_0,$$

where $|y_0| \leqslant u_0$, is $[t_0, t_0 + a]$.

However, so far there seems to be also no stochastic version of Wintner theorem for SFDEs so much as FDEs. Therefore, our another aim is to extend Wintner theorem from ODEs to SFDEs and obtain the global solutions of (5) and (6).

This paper is organized as follows. We firstly obtain the local existence–uniqueness of solutions of (5) and (6) by employing the ordinary local Lipschitz condition and Picard sequence. Furthermore, a continuation theorem for the SFDE (5) with the initial condition (6) is given and it is a generalization of the continuation theorem for FDEs in [21] and [22]. The key of its proof is to deal with the complexity brought by the various sample paths. To overcome this difficult, we construct an especial subset generated by all sample paths with explosion in the sample space and derive a contradiction by the indicator function for this subset and using the analogous methods from FDEs (see [21]) if the continuation theorem is not true. Finally, we extend Wintner theorem to SFDEs by establishing some powerful differential inequalities of continuous functions with Dini derivative. Furthermore, we obtain some useful corollaries ensuring the global existence–uniqueness of solutions of the SFDE (5) and (6), which extend and improve the global existence–uniqueness theorems of Driver for FDEs [21] and Taniguchi for SODEs [7]. The main methods used in the proofs of the theorems are motivated by the papers [7,8,18,19,21,22].

2. Preliminaries

In this section, we introduce some notations and recall some basic definitions.

For $A, B \in R^{m\times n}$ or $A, B \in R^n$, $A \geqslant B$ ($A \leqslant B$, $A > B$, $A < B$) means that each pair of corresponding elements of A and B satisfies the inequality "$\geqslant$ ($\leqslant$, $>$, $<$)."

$C(X, Y)$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, let $C \triangleq C([-\tau, 0], R^n)$ with the norm $\|\varphi\| = \max_{-\tau\leqslant s\leqslant 0} |\varphi(s)|$, where $\tau > 0$ and $|\,.\,|$ is any norm in R^n.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant t_0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets in $\mathcal{F}$). $w(t) = (w_1(t), \dots, w_m(t))^T$ is an m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant t_0}, P)$.

The following discussions in L^2 are valid for L^p. We shall adopt the usual manner (see [17–19]) and let $L^2(\Omega, C(J, R^n))$ be the space of ($\mathcal{F}$, Borel C)-measurable maps $\Omega \to C(J, R^n)$ which are L^2 in the Bochner sense. Give $L^2(\Omega, C(J, R^n))$ the norm

$$\|\xi\|_{\Omega J} = \left[\int_\Omega \max_{t\in J}\bigl|\xi(t,\omega)\bigr|^2\, dP(\omega)\right]^{\frac{1}{2}} = \left[\mathbb{E}\max_{t\in J}\bigl|\xi(t,\omega)\bigr|^2\right]^{\frac{1}{2}},$$

where $J \subset R$ is a bounded interval. Especially, when $J = [-\tau, 0]$,

$$L^2\bigl(\Omega, C\bigl(J, R^n\bigr)\bigr) = L^2(\Omega, C).$$

For convenience, we denote the norm of $\xi \in L^2(\Omega, C)$ by

$$\|\xi\|_\Omega \triangleq \|\xi\|_{\Omega[-\tau,0]} = \bigl[\mathbb{E}\|\xi\|^2\bigr]^{\frac{1}{2}}.$$

Let $L^2_D(\Omega, C([t_0 - \tau, a], R^n))$ be the space of all processes $x(t) \in L^2(\Omega, C([t_0 - \tau, a], R^n))$ such that $x(t)$ is $\mathcal{F}_{t_0}$-measurable for all $t \in [t_0 - \tau, t_0]$ and $x(t)$ is $\mathcal{F}_t$-measurable for all $t \in [t_0, a]$. Then, $L^2_D(\Omega, C([t_0 - \tau, a], R^n))$ is a closed linear subspace of $L^2(\Omega, C([t_0 - \tau, a], R^n))$ [18, p. 31].

For $\xi \in L^2(\Omega, C)$ and $r > 0$, we denote

$$S(\xi, r) = \bigl\{\phi \in L^2(\Omega, C){:}\ \|\phi - \xi\|_\Omega \leqslant r\bigr\}.$$

For Banach space $L^2(\Omega, R^n)$, we define the norm

$$|x|_\Omega \triangleq \bigl(\mathbb{E}|x|^2\bigr)^{\frac{1}{2}}.$$

In this paper, we also employ $|\cdot|_\Omega$ to denote the norm of Banach space $L^2(\Omega, R^{n\times m})$.

Throughout this paper, we suppose $\xi \in L^2(\Omega, C)$ in (6) is an $\mathcal{F}_{t_0}$-measurable process and for (5), the drift coefficient function

$$B : [t_0, T) \times L^2(\Omega, C) \to L^2\bigl(\Omega, R^n\bigr),$$

and the diffusion coefficient

$$\sigma : [t_0, T) \times L^2(\Omega, C) \to L^2\bigl(\Omega, R^{n\times m}\bigr),$$

satisfy the following condition:

(Q) B and σ are continuous on $[t_0, T) \times L^2(\Omega, C)$ and for each $\mathcal{F}_t$-adapted process $y_t : [t_0, T) \to L^2(\Omega, C)$, the processes $B(t, y_t)$ and $\sigma(t, y_t)$ are also $\mathcal{F}_t$-adapted.

Definition 1. Let $\bar{J} = [t_0 - \tau, a)$ or $\bar{J} = [t_0 - \tau, a]$, where $t_0 < a \leqslant T$. R^n-value stochastic process $x(t)$ defined on $\bar{J}$ is called a solution of (5) and (6) if $x(t) \in L^2_D(\Omega, C(\bar{J}, R^n))$ and satisfies (5) and (6) almost surely. The solution $x(t)$ of (5) and (6) on interval $\bar{J}$ is said to be unique if any other solution $\bar{x}(t)$ on interval $\bar{J}$ is indistinguishable from it, that is,

$$P\{x(t) = \bar{x}(t) \text{ for all } t \in \bar{J}\} = 1.$$

Definition 2. Let $x(t)$ on J_1 and $\bar{x}(t)$ on J_2 both be solutions of (5) and (6). If $J_1 \subset J_2$, $J_1 \neq J_2$ and $P\{x(t) = \bar{x}(t) \text{ for all } t \in J_1\} = 1$, we say $\bar{x}(t)$ is a continuation of $x(t)$, or $x(t)$ can be continued to J_2. A solution $x(t)$ is non-continuable if it has no continuation. The existing interval of non-continuable solution $x(t)$ is called the maximum existing interval of $x(t)$.

Definition 3. It is said that a sample path $x(t, \omega)$ explodes in $[t_0 - \tau, T)$ if for any integer $k > 0$, there exists a time $s \in [t_0 - \tau, T)$ such that $|x(s, \omega)| \geqslant k$. And the solution $x(t, \omega)$ of (5) and (6) explodes in $[t_0 - \tau, T)$ if there exists a measurable subset $S \subset \Omega$ with $P(S) > 0$ such that the sample path $x(t, \omega)$ explodes in $[t_0 - \tau, T)$ for almost all $\omega \in S$.

Definition 4. The functional $F : [t_0, T) \times L^2(\Omega, C) \to L^2(\Omega, R^n)$ is said to be quasi-bounded if for any constants $\beta \in (t_0, T)$ and $\alpha > 0$, there exists a positive constant M such that

$$\big|F(t, \phi)\big|_\Omega \leqslant M,$$

provided that

$$t \in [t_0, \beta] \quad \text{and} \quad \|\phi\|_\Omega \leqslant \alpha.$$

Definition 5. The functional $F : [t_0, T) \times L^2(\Omega, C) \to L^2(\Omega, R^n)$ is said to satisfy the local Lipschitz condition at point $(\bar{t}_0, \xi)$ if there exist positive constants b, r and K such that

$$\big|F(t, \phi) - F(t, \psi)\big|_\Omega \leqslant K\|\phi - \psi\|_\Omega,$$

for all $t \in [\bar{t}_0 - b, \bar{t}_0 + b] \cap [t_0, T)$ and $\phi, \psi \in S(\xi, r)$. Moreover, F is said to satisfy the local Lipschitz condition in the region $[t_0, T) \times L^2(\Omega, C)$ if F satisfies the local Lipschitz condition for any point $(\bar{t}, \bar{\xi}) \in [t_0, T) \times L^2(\Omega, C)$.

Definition 6. A function $h \in C(R \times R^m \times R^m, R^m)$ is called an H_m-function if for any $t \geqslant t_0 \in R$ and any $u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)} \in R^m$, every ith element of h satisfies $h_i(t, u^{(1)}, v^{(1)}) \leqslant h_i(t, u^{(2)}, v^{(2)})$ when $u^{(1)} \leqslant u^{(2)}$ with $u_i^{(1)} = u_i^{(2)}$ and $v^{(1)} \leqslant v^{(2)}$.

3. Local existence–uniqueness theorem

In order to obtain the local existence and the uniqueness of solutions of the SFDE (5) with the initial condition (6), define $x_{t_0}^0 = \xi$ and $x^0(t) = \xi(0)$, for $t \in [t_0, T)$. Let $x_{t_0}^n = \xi$, $n = 1, 2, \ldots,$ and define Picard sequence

$$x^n(t) = \xi(0) + \int_{t_0}^{t} B\bigl(s, x_s^{n-1}\bigr)\,ds + \int_{t_0}^{t} \sigma\bigl(s, x_s^{n-1}\bigr)\,d\omega(s), \quad t \in [t_0, T),\ n = 1, 2, \ldots.$$

Lemma 1. *Assume B and σ satisfy the condition* (Q) *and the local Lipschitz condition at $(t_0, \xi) \in [t_0, T) \times L^2(\Omega, C)$, then there exists a $t_1 \in (t_0, T)$ such that*

$$x^n(t) \in L_D^2\bigl(\Omega, C\bigl([t_0 - \tau, t_1], R^n\bigr)\bigr), \quad n = 0, 1, \ldots, \tag{10}$$

$$\text{for each } t \in [t_0, t_1], \quad x_t^n \in L^2(\Omega, C), \quad n = 0, 1, \ldots, \tag{11}$$

$$\bigl\|x^n(t) - \xi(0)\bigr\|_{\Omega[t_0, t_1]} \leqslant \frac{r_1}{2}, \quad n = 0, 1, \ldots, \tag{12}$$

$$\bigl\|x_t^n - \xi\bigr\|_{\Omega} \leqslant r_1, \quad t \in [t_0, t_1],\ n = 0, 1, \ldots, \tag{13}$$

where r_1 is a positive constant.

Proof. Since B and σ satisfy the local Lipschitz condition at $(t_0, \xi) \in [t_0, T) \times L^2(\Omega, C)$, then there exist positive constants K_1, r_1 and b_1 with $t_0 + b_1 < T$ such that

$$\bigl|B(t, \phi) - B(t, \psi)\bigr|_{\Omega} \leqslant K_1 \|\phi - \psi\|_{\Omega}, \tag{14}$$

$$\bigl|\sigma(t, \phi) - \sigma(t, \psi)\bigr|_{\Omega} \leqslant K_1 \|\phi - \psi\|_{\Omega}, \tag{15}$$

for all $t \in [t_0, t_0 + b_1]$ and all $\phi, \psi \in S(\xi, r_1)$. Since B and σ are continuous on $[t_0, T) \times L^2(\Omega, C)$, $|B(t, \xi)|_{\Omega}$ and $|\sigma(t, \xi)|_{\Omega}$ are continuous on t for the above given $\xi \in L^2(\Omega, C)$. So, there exists a positive constant M such that

$$\bigl|B(t, \xi)\bigr|_{\Omega} \leqslant M, \qquad \bigl|\sigma(t, \xi)\bigr|_{\Omega} \leqslant M, \quad \text{for all } t \in [t_0, t_0 + b_1]. \tag{16}$$

Choose $t^* \in (t_0, t_0 + b_1]$ satisfying

$$4(b_1 + 4)\bigl(K_1^2 r_1^2 + M^2\bigr)\bigl(t^* - t_0\bigr) \leqslant \frac{r_1^2}{4}. \tag{17}$$

Furthermore, from the definition of x_t^0, there exists a $t_1 \in (t_0, t^*]$ such that

$$\bigl\|x_t^0 - \xi\bigr\|_{\Omega} = \bigl\|x_t^0 - x_{t_0}^0\bigr\|_{\Omega} \leqslant \frac{r_1}{2}, \quad t \in [t_0, t_1]. \tag{18}$$

Now, let us prove Lemma 1 by the mathematical induction. It follows from the definition of $x^0(t)$ and (18) that (10)–(13) hold for $n = 0$. Now, for $n = k$, suppose (10)–(13) hold. Then for $n = k + 1$, by Schwarz inequality, Doob's martingale inequality and (14)–(17), we can obtain

$$\begin{aligned}
\left\|x^{k+1}(t)-\xi(0)\right\|^2_{\Omega[t_0,t_1]} &= \mathbb{E}\sup_{t_0\leqslant t\leqslant t_1}\left|\int_{t_0}^{t} B\left(s,x_s^k\right)ds+\int_{t_0}^{t}\sigma\left(s,x_s^k\right)d\omega(s)\right|^2 \\
&\leqslant 2\mathbb{E}\sup_{t_0\leqslant t\leqslant t_1}\left[\left|\int_{t_0}^{t} B\left(s,x_s^k\right)ds\right|^2+\left|\int_{t_0}^{t}\sigma\left(s,x_s^k\right)d\omega(s)\right|^2\right] \\
&\leqslant 2(t_1-t_0)\int_{t_0}^{t_1}\mathbb{E}\left|B\left(s,x_s^k\right)\right|^2ds+8\int_{t_0}^{t_1}\mathbb{E}\left|\sigma\left(s,x_s^k\right)\right|^2ds \\
&\leqslant 4b_1\int_{t_0}^{t_1}\left[\left|B\left(s,x_s^k\right)-B(s,\xi)\right|^2_\Omega+\left|B(s,\xi)\right|^2_\Omega\right]ds \\
&\quad +16\int_{t_0}^{t_1}\left[\left|\sigma\left(s,x_s^k\right)-\sigma(s,\xi)\right|^2_\Omega+\left|\sigma(s,\xi)\right|^2_\Omega\right]ds \\
&\leqslant 4(b_1+4)\int_{t_0}^{t_1}\left[K_1^2\left\|x_s^k-\xi\right\|^2_\Omega+M^2\right]ds \\
&\leqslant 4(b_1+4)\left(K_1^2r_1^2+M^2\right)(t_1-t_0) \\
&\leqslant \frac{r_1^2}{4}.
\end{aligned} \tag{19}$$

Therefore, by the definition of the sequence $\{x^{k+1}(t)\}$ on $[t_0-\tau,T)$, (18) and (19),

$$\begin{aligned}
\left\|x_t^{k+1}-\xi\right\|^2_\Omega &= \mathbb{E}\max_{-\tau\leqslant s\leqslant 0}\left|x^{k+1}(t+s)-\xi(s)\right|^2 \\
&= \mathbb{E}\max_{-\tau\leqslant s\leqslant 0}\left|x^{k+1}(t+s)-x^0(t+s)+x^0(t+s)-\xi(s)\right|^2 \\
&\leqslant 2\mathbb{E}\max_{-\tau\leqslant s\leqslant 0}\left[\left|x^{k+1}(t+s)-x^0(t+s)\right|^2+\left|x^0(t+s)-\xi(s)\right|^2\right] \\
&\leqslant 2\mathbb{E}\left[\max_{-\tau\leqslant s\leqslant 0}\left|x^{k+1}(t+s)-x^0(t+s)\right|^2+\max_{-\tau\leqslant s\leqslant 0}\left|x^0(t+s)-\xi(s)\right|^2\right] \\
&= 2\mathbb{E}\max_{\substack{-\tau\leqslant s\leqslant 0\\ t+s\geqslant t_0}}\left|x^{k+1}(t+s)-x^0(t+s)\right|^2+2\mathbb{E}\left\|x_t^0-\xi\right\|^2 \\
&= 2\mathbb{E}\max_{\substack{-\tau\leqslant s\leqslant 0\\ t+s\geqslant t_0}}\left|x^{k+1}(t+s)-\xi(0)\right|^2+2\left\|x_t^0-\xi\right\|^2_\Omega \\
&\leqslant 2\left\|x^{k+1}(t)-\xi(0)\right\|^2_{\Omega[t_0,t_1]}+\frac{r_1^2}{2} \\
&\leqslant r_1^2, \quad t\in[t_0,t_1].
\end{aligned} \tag{20}$$

Consequently, (19) and (20) assure that (12)–(13) are true for $n = k + 1$. Furthermore, (10) and (11) hold for $n = k + 1$ by the condition (Q). Hence, by the mathematical induction, (10)–(13) hold for every integer $n \geqslant 0$. The proof is completed. □

Theorem 1 *(Local existence–uniqueness theorem). Assume B and σ satisfy the condition* (Q) *and the local Lipschitz condition at $(t_0, \xi) \in [t_0, T) \times L^2(\Omega, C)$, then there is a $t_1 \in (t_0, T)$ such that the SFDE* (5) *with the initial condition* (6) *has a unique solution on* $[t_0 - \tau, t_1]$.

Proof. Since B and σ satisfy the local Lipschitz condition at $(t_0, \xi) \in [t_0, T) \times L^2(\Omega, C)$, there exist positive constants b_1, r_1 and K_1 such that

$$\big|B(t, \phi) - B(t, \psi)\big|_\Omega \leqslant K_1 \|\phi - \psi\|_\Omega, \tag{21}$$

$$\big|\sigma(t, \phi) - \sigma(t, \psi)\big|_\Omega \leqslant K_1 \|\phi - \psi\|_\Omega, \tag{22}$$

for all $t \in [t_0, t_0 + b_1]$ and $\phi, \psi \in S(\xi, r_1)$. For the above constant r_1, by Lemma 1, there exists a $t_1 \in (t_0, T)$ such that (10)–(13) hold.

Now, by induction, we will derive that

$$\big\|x^{n+1}(s) - x^n(s)\big\|^2_{\Omega[t_0,t]} \leqslant \frac{r_1^2[\bar{M}(t - t_0)]^n}{4 \times n!}, \quad t \in [t_0, t_1], \tag{23}$$

where $\bar{M} = 2(t_1 - t_0 + 4)K_1^2$, $n = 0, 1, \ldots.$

From (12), we have

$$\begin{aligned}
\big\|x^1(s) - x^0(s)\big\|^2_{\Omega[t_0,t]} &\leqslant \big\|x^1(s) - x^0(s)\big\|^2_{\Omega[t_0,t_1]} \\
&= \big\|x^1(s) - \xi(0)\big\|^2_{\Omega[t_0,t_1]} \\
&\leqslant \frac{r_1^2}{4}, \quad t \in [t_0, t_1],
\end{aligned}$$

yielding the inequality (23) holds for $n = 0$. Now we suppose that (23) holds for $n = k$. Then, by Schwarz inequality, Doob's martingale inequality and (13), (21) and (22), for $n = k + 1$, we have

$$\begin{aligned}
&\big\|x^{k+2}(s) - x^{k+1}(s)\big\|^2_{\Omega[t_0,t]} \\
&\quad = \mathbb{E} \sup_{t_0 \leqslant s \leqslant t} \left| \int_{t_0}^{s} \big[B\big(s, x_s^{k+1}\big) - B\big(s, x_s^k\big)\big]\, ds + \int_{t_0}^{s} \sigma\big(s, x_s^{k+1}\big) - \sigma\big(s, x_s^k\big)\, d\omega(s) \right|^2 \\
&\quad \leqslant 2\mathbb{E} \sup_{t_0 \leqslant s \leqslant t} \left[\left| \int_{t_0}^{s} \big[B\big(s, x_s^{k+1}\big) - B\big(s, x_s^k\big)\big]\, ds \right|^2 + \left| \int_{t_0}^{s} \sigma\big(s, x_s^{k+1}\big) - \sigma\big(s, x_s^k\big)\, d\omega(s) \right|^2 \right] \\
&\quad \leqslant 2\mathbb{E} \left[\sup_{t_0 \leqslant s \leqslant t} \left| \int_{t_0}^{s} \big[B\big(s, x_s^{k+1}\big) - B\big(s, x_s^k\big)\big]\, ds \right|^2 + \sup_{t_0 \leqslant s \leqslant t} \left| \int_{t_0}^{s} \sigma\big(s, x_s^{k+1}\big) - \sigma\big(s, x_s^k\big)\, d\omega(s) \right|^2 \right]
\end{aligned}$$

$$\leqslant 2(t_1-t_0)\mathbb{E}\int_{t_0}^{t}\left[B\left(s,x_s^{k+1}\right)-B\left(s,x_s^{k}\right)\right]^2 ds+8\mathbb{E}\int_{t_0}^{t}\left[\sigma\left(s,x_s^{k+1}\right)-\sigma\left(s,x_s^{k}\right)\right]^2 ds$$

$$\leqslant 2(t_1-t_0+4)K_1^2\int_{t_0}^{t}\left\|x_s^{k+1}-x_s^{k}\right\|_{\Omega}^2 ds$$

$$=\bar{M}\int_{t_0}^{t}\mathbb{E}\sup_{-\tau\leqslant\varrho\leqslant 0}\left|x^{k+1}(s+\varrho)-x^{k}(s+\varrho)\right|^2 ds$$

$$=\bar{M}\int_{t_0}^{t}\mathbb{E}\sup_{t_0\leqslant\nu\leqslant s}\left|x^{k+1}(\nu)-x^{k}(\nu)\right|^2 ds$$

$$=\bar{M}\int_{t_0}^{t}\left\|x^{k+1}(\nu)-x^{k}(\nu)\right\|_{\Omega[t_0,s]}^2 ds$$

$$\leqslant \bar{M}\int_{t_0}^{t}\frac{r_1^2[\bar{M}(s-t_0)]^k}{4\times k!}\,ds$$

$$=\frac{r_1^2[\bar{M}(t-t_0)]^{k+1}}{4\times(k+1)!},\quad t\in[t_0,t_1]. \tag{24}$$

Thus, the inequality (23) holds. So, we have

$$\left\|x^{n+1}(t)-x^{n}(t)\right\|_{\Omega[t_0-\tau,t_1]}^2=\left\|x^{n+1}(t)-x^{n}(t)\right\|_{\Omega[t_0,t_1]}^2\leqslant\frac{r_1^2[\bar{M}(t_1-t_0)]^n}{4\times n!},\quad n=0,1,\ldots.$$

Since $L_D^2(\Omega,C([t_0-\tau,t_1],R^n))$ is a closed linear subspace of $L^2(\Omega,C([t_0-\tau,t_1],R^n))$, the series $\{x^n(t)\}$ converge to some $x(t)\in L_D^2(\Omega,C([t_0-\tau,t_1],R^n))$.

Next, we shall show the stochastic process $x(t)$ is a local solution of the SFDE (5) with the initial condition (6). By the same ways in (24), we obtain

$$\left\|x^n(t)-\left[\xi(0)+\int_{t_0}^{t}B(s,x_s)\,ds+\int_{t_0}^{t}\sigma(s,x_s)\,d\omega(s)\right]\right\|_{\Omega[t_0,t_1]}^2$$

$$=\mathbb{E}\sup_{t_0\leqslant t\leqslant t_1}\left|\int_{t_0}^{t}\left[B\left(s,x_s^{n-1}\right)-B(s,x_s)\right]ds+\int_{t_0}^{t}\left[\sigma\left(s,x_s^{n-1}\right)-\sigma(s,x_s)\right]d\omega(s)\right|^2$$

$$\leqslant \bar{M}\int_{t_0}^{t_1}\left\|x^{n-1}(\nu)-x(\nu)\right\|_{\Omega[t_0,s]}^2 ds$$

$$\leqslant \bar{M} \int_{t_0}^{t_1} \left\| x^{n-1}(\nu) - x(\nu) \right\|^2_{\Omega[t_0,t_1]} ds$$
$$= \bar{M}(t_1 - t_0) \left\| x^{n-1}(\nu) - x(\nu) \right\|^2_{\Omega[t_0,t_1]}$$
$$\to 0, \quad n \to \infty.$$

That is,

$$x(t) = \xi(0) + \int_{t_0}^{t} B(s, x_s)\, ds + \int_{t_0}^{t} \sigma(s, x_s)\, d\omega(s). \tag{25}$$

The above expression demonstrate that $x(t)$ is a solution of the SFDE (5) with the initial condition (6).

Finally, we shall show the uniqueness of the solutions of the initial value problem (5) and (6). Let $x(t)$ and $y(t)$ be any two solutions of (5) and (6). By the same ways in (24), we obtain

$$\left\| x(s) - y(s) \right\|^2_{\Omega[t_0,t]} \leqslant \bar{M} \int_{t_0}^{t} \left\| x(\nu) - y(\nu) \right\|^2_{\Omega[t_0,s]} ds, \quad t \in [t_0, t_1].$$

Applying the Gronwall inequality to yield

$$\left\| x(s) - y(s) \right\|^2_{\Omega[t_0,t]} = 0, \quad t \in [t_0, t_1].$$

The above expression means that

$$P\left\{ x(t) = y(t), \text{ for all } t \in [t_0 - \tau, t_1] \right\} = 1.$$

Thus, the proof is completed. □

Remark 1. Theorem 1 is a natural generalization of the local existence and uniqueness theorem [21, Theorem A, p. 301] of the functional differential equation

$$\dot{x}(t) = f(t, x_t), \qquad x_{t_0} = \phi \in C, \tag{26}$$

where $x \in R^n$ and $f \in C([t_0, \infty) \times C, R^n)$.

4. Continuation theorem

In this section, we present the following continuation theorem for the initial value problem (5) and (6).

Theorem 2 *(Continuation theorem). Assume that B and σ are quasi-bounded, satisfy the condition* (Q) *and the local Lipschitz condition in* $[t_0, T) \times L^2(\Omega, C)$, *the following conclusions are true.*

(I) *The SFDE* (5) *with initial condition* (6) *has a unique non-continuable solution* $x(t)$, *whose maximum existing interval is assumed to be* $[t_0 - \tau, \beta_1)$.

(II) *For every closed bounded set* $A \subset [t_0 - \tau, T) \times R^n$,

$$P\big\{\big(t, x(t)\big) \notin A,\ \textit{for some } t \in [t_0, T)\big\} = 1. \tag{27}$$

Proof. From Theorem 1, the SFDE (5) with initial condition (6) has a unique solution $x(t) \in L^2_D(\Omega, C([t_0 - \tau, t_1], R^n))$. Note that $x_{t_1} \in L^2(\Omega, C)$ and B, σ satisfy the local Lipschitz condition in $[t_0, T) \times L^2(\Omega, C)$. Thus applying Theorem 1 to the SFDE (5) with the initial condition (t_1, x_{t_1}), the solution $x(t)$ of (5) and (6) can be continued to $[t_0 - \tau, t_1 + \delta_1]$, where δ_1 is a positive constant satisfying $t_1 + \delta_1 < T$. Furthermore, $x(t)$ is the unique solution of (5) and (6) on $[t_0 - \tau, t_1 + \delta_1]$. Repeat the above procedure and define

$$\beta_1 = \sup\big\{s \in R\colon\ x(t) \text{ can be continued to } [t_0 - \tau, s]\big\}.$$

Then $\beta_1 \in (t_0, T]$, $x(t)$ is the unique non-continuable solution of the initial value problem (5) and (6) and its maximum existing interval is $[t_0 - \tau, \beta_1)$. Obviously, its maximum existing interval must not be $[t_0 - \tau, \beta_1]$ by the same continuation way of $x(t)$ at $t = t_1$.

Then, the proof of (I) is completed.

In (II), the case $\beta_1 = T$ is trivial. So we suppose $\beta_1 < T$. If the conclusion of (II) is not true, there must exist a closed bounded set $A_1 \subset [t_0 - \tau, T) \times R^n$ such that

$$P\big\{\big(t, x(t)\big) \notin A_1,\ \text{for some } t \in [t_0, T)\big\} < 1.$$

Denote

$$\Omega_1 = \big\{\omega \in \Omega\colon\ \big(t, x(t)\big) \notin A_1,\ \text{for some } t \in [t_0, T)\big\}, \qquad \Omega_2 = \Omega \setminus \Omega_1.$$

Then, $P(\Omega_1) < 1$ and $P(\Omega_2) = 1 - P(\Omega_1) > 0$.

Let

$$\hat{x}(t) = \begin{cases} x(t), & \omega \in \Omega_2, \\ \xi(0), & \omega \notin \Omega_2. \end{cases} \tag{28}$$

Then $\hat{x}(t)$ satisfies

$$\begin{cases} x(t) = \xi(0) + \displaystyle\int_{t_0}^{t} I_{\Omega_2} B(s, x_s)\, ds + \int_{t_0}^{t} I_{\Omega_2} \sigma(s, x_s)\, d\omega(s), \\ x_{t_0} = \xi, \end{cases} \tag{29}$$

where I_{Ω_2} denotes the indicator function (see [7]) for Ω_2. On the other hand, by (I), the initial value problem (29) has a unique non-continuable solution. So, by uniqueness, $\hat{x}(t)$ is the unique non-continuable solution of (29). Let its maximum existing interval be $[t_0 - \tau, \beta_2)$. By the definitions of Ω_2, we obtain

$$\big(t, \hat{x}(t)\big) \in A_1, \quad \text{for all } t \in [t_0, T),\ \omega \in \Omega_2. \tag{30}$$

Since $A_1 \subset [t_0-\tau, T)\times R^n$ is a closed bounded set, we obtain that $\beta_2 \in [\beta_1, T)$ is finite. From the boundedness of A_1 and (30), there exists a constant $\alpha_1 > \|\xi\|_\Omega$ such that

$$\left|\hat{x}(t)\right| < \alpha_1, \quad \text{for all } t \in [t_0, \beta_2),\ \omega \in \Omega_2.$$

Thus, from (28), we have

$$\|\hat{x}_t\|_\Omega < \alpha_1, \quad \text{for all } t \in [t_0, \beta_2).$$

By the quasi-boundedness of B and σ, there is a positive constant μ_1 such that

$$\left|B(t,\hat{x}_t)\right|_\Omega \leqslant \mu_1, \qquad \left|\sigma(t,\hat{x}_t)\right|_\Omega \leqslant \mu_1, \quad \text{for all } t \in [t_0, \beta_2). \tag{31}$$

By using the properties of Brownian motion, (29), (31) and Schwarz inequality, we obtain for all $\hat{t}_1, \hat{t}_2 \in [t_0, \beta_2)$,

$$\begin{aligned}
\left|\hat{x}(\hat{t}_1)-\hat{x}(\hat{t}_2)\right|_\Omega^2 &= \mathbb{E}\left|\hat{x}(\hat{t}_1)-\hat{x}(\hat{t}_2)\right|^2 \\
&\leqslant 2\mathbb{E}\left|\int_{\hat{t}_1}^{\hat{t}_2} I_{\Omega_2}B(s,\hat{x}_s)\,ds\right|^2 + 2\mathbb{E}\left|\int_{\hat{t}_1}^{\hat{t}_2} I_{\Omega_2}\sigma(s,\hat{x}_s)\,d\omega(s)\right|^2 \\
&\leqslant 2|\hat{t}_2-\hat{t}_1|\int_{\hat{t}_1}^{\hat{t}_2}\left|I_{\Omega_2}B(s,\hat{x}_s)\right|_\Omega^2 ds + 2\int_{\hat{t}_1}^{\hat{t}_2}\left|I_{\Omega_2}\sigma(s,\hat{x}_s)\right|_\Omega^2 ds \\
&\leqslant 2|\hat{t}_2-\hat{t}_1|\int_{\hat{t}_1}^{\hat{t}_2}\left|B(s,\hat{x}_s)\right|_\Omega^2 ds + 2\int_{\hat{t}_1}^{\hat{t}_2}\left|\sigma(s,\hat{x}_s)\right|_\Omega^2 ds \\
&\leqslant 2\mu_1^2(\beta_2-t_0+1)|\hat{t}_2-\hat{t}_1|.
\end{aligned} \tag{32}$$

Then, there exists a $\bar{\xi} \in L^2(\Omega, R^n)$ such that

$$\lim_{t\to\beta_2^-}\hat{x}(t) = \bar{\xi}.$$

Let

$$\tilde{x}(t) = \begin{cases} \hat{x}(t), & t \in [t_0-\tau,\ \beta_2), \\ \bar{\xi}, & t = \beta_2. \end{cases}$$

Then $\tilde{x}(t)$ is the unique solution of the initial value problem (29). So, $\hat{x}(t)$ can be continued to $[t_0-\tau, \beta_2]$. This contradicts the fact that the maximum existing interval of $\hat{x}(t)$ is $[t_0-\tau, \beta_2)$. So, (II) is true.

Then, the proof of Theorem 2 is completed. □

Remark 2. For the FDE (26), (27) means that

$$\big(t, x(t)\big) \notin A, \quad \text{for some } t \in [t_0, \beta_1).$$

So, Theorem 2 is a natural generalization of the continuation theorems (see [21, Theorem C, p. 306] and [22, Theorem 3.2, p. 46]) of the FDE (26).

Corollary 1. *With the same conditions in Theorem* 2*, the following conclusions are true.*

(I) *If* $\beta_1 < T$*, the solution* $x(t)$ *of the SFDE* (5) *with the initial condition* (6) *explodes in* $[t_0 - \tau, \beta_1)$.
(II) *If the solution* $x(t)$ *of* (5) *and* (6) *is bounded,* $x(t)$ *exists on* $[t_0 - \tau, T)$.

Proof. If $\beta_1 < T$, we denote

$$\Omega_3 = \big\{\omega \in \Omega\colon\ x(t) \text{ exist on } [t_0 - \tau, \beta_1) \text{ and } x(\beta_1) \text{ does not exist}\big\}.$$

Then, $P(\Omega_3) > 0$. Or else,

$$x(\beta_1) \quad \text{exists for a.e. } \omega \in \Omega,$$

which contradicts the fact that the maximum existing interval of $x(t)$ is $[t_0 - \tau, \beta_1)$.

For every integer $N > 0$, choose closed bounded sets

$$A_N = \big\{z = (z_1, \ldots, z_n)^T \in R^n\colon\ |z_i| \leqslant N,\ i = 1, 2, \ldots, n\big\},$$
$$\bar{A}_N = [t_0 - \tau, \beta_1] \times A_N.$$

By (II) of Theorem 2,

$$P\big\{\big(t, x(t)\big) \notin \bar{A}_N,\ \text{for some } t \in [t_0, T)\big\} = 1.$$

That is, for a.e. $\omega \in \Omega_3$,

$$\big(t, x(t)\big) \notin \bar{A}_N, \quad \text{for some } t \in [t_0, T).$$

By the definition of Ω_3, we have, for a.e. $\omega \in \Omega_3$,

$$x(t) \notin A_N, \quad \text{for some } t \in [t_0, \beta_1).$$

That is, for a.e. $\omega \in \Omega_3$,

$$\big|x(t)\big| > N, \quad \text{for some } t \in [t_0, \beta_1). \tag{33}$$

This, together with $P(\Omega_3) > 0$, yields that the solution $x(t)$ of (5) and (6) explodes in $[t_0 - \tau, \beta_1)$. This completes the proof of (I).

(II) holds obviously by (I). Then, the proof of Corollary 1 is completed. □

5. Global existence theorems

In this section, we first establish a delay differential inequality. Then by using this inequality, properties of H_m-functions and Theorem 2, we obtain the global existence of solutions of (5) and (6).

For the vector functions $x(t) = (x_1(t), \dots, x_m(t))^T \in C(R, R^m)$, we denote

$$\bar{x}(t) = \big(\bar{x}_1(t), \dots, \bar{x}_m(t)\big)^T, \qquad \bar{x}_i(t) = \sup_{-\tau \leqslant s \leqslant 0} x_i(t+s), \quad i = 1, 2, \dots, m,$$

and define the Dini upper right derivative as follows:

$$D^+x(t) = \big(D^+x_1(t), \dots, D^+x_m(t)\big)^T, \qquad D^+x_i(t) = \limsup_{h \to 0^+} \frac{x_i(t+h) - x_i(t)}{h},$$
$$i = 1, 2, \dots, m.$$

Lemma 2. *Let h be an H_m-function, $x(t)$ and $y(t)$ be continuous and satisfy*

$$x(t) \leqslant y(t), \quad t \in [t_0 - \tau, t_0].$$

Furthermore, $x(t)$ is a solution of

$$D^+x(t) \leqslant h\big(t, x(t), \bar{x}(t)\big), \quad t \geqslant t_0, \tag{34}$$

$y(t)$ is a solution of

$$\dot{y}(t) = h\big(t, y(t), \bar{y}(t)\big), \quad t \geqslant t_0.$$

Then for all $t \geqslant t_0$,

$$x(t) \leqslant y(t). \tag{35}$$

Proof. For any positive constant $\varepsilon > 0$, denote by $y^\varepsilon(t)$ the solution of the delay differential equation

$$\dot{y}^\varepsilon(t) = h\big(t, y^\varepsilon(t), \bar{y}^\varepsilon(t)\big) + \varepsilon, \quad t \geqslant t_0,$$

with the initial condition $y^\varepsilon(t) = y(t)$, $\forall t \in [t_0 - \tau, t_0]$. We at first shall prove that

$$x(t) \leqslant y^\varepsilon(t), \quad \forall t \geqslant t_0. \tag{36}$$

If the inequality (36) is not true, then there must be a $t^* \geqslant t_0$, some integer k and a positive constant δ such that

$$x_k\big(t^*\big) = y_k^\varepsilon\big(t^*\big), \qquad x_k(t) > y_k^\varepsilon(t), \quad \forall t \in \big[t^*, t^* + \delta\big], \tag{37}$$

$$x_i(t) \leqslant y_i^\varepsilon(t), \quad t \in \big[t_0 - \tau, t^*\big],\ i = 1, \dots, m. \tag{38}$$

Then, we must have

$$D^+x_k\big(t^*\big) \geqslant \dot{y}_k^{\varepsilon}\big(t^*\big). \tag{39}$$

On the other hand, from the equality in (37), (38) and properties of H_m-function h, the inequality (34) implies that

$$\begin{aligned} D^+x_k\big(t^*\big) &\leqslant h_k\big(t^*, x\big(t^*\big), \bar{x}\big(t^*\big)\big) \leqslant h_k\big(t^*, y^{\varepsilon}\big(t^*\big), \bar{y}^{\varepsilon}\big(t^*\big)\big) \\ &< h_k\big(t^*, y^{\varepsilon}\big(t^*\big), \bar{y}^{\varepsilon}\big(t^*\big)\big) + \varepsilon = \dot{y}_k^{\varepsilon}\big(t^*\big), \end{aligned}$$

which contradicts the inequality (39). Thus the inequality (36) holds. By the continuous dependence theorem [22, Theorem 2.2] of functional differential equations, we have

$$y^{\varepsilon}(t) \to y(t), \quad \text{when } \varepsilon \to 0.$$

So the inequality (35) holds. The proof is completed. □

Remark 3. Actually, our Lemma 2 is natural generalization of Lemma 8.2 in [23, p. 72]. Here, we use the Dini derivative to replace ordinary derivative and use the continuous dependence theorem to replace the classical method using a family of functions. Lemma 2 also improves and extends some known results, for example, Theorem 1 of [24] when $x(t)$ and $y(t)$ are differentiable, Lemma 1 of [25] when the inequality (34) is strict. By the way, there is a gap in the proof of Lemma 1 in [25] since they used an insufficient fact that the upper right derivative $D^+x(t^*) \geqslant 0$ if $x(t)$ is increasing on the left side of t^*.

Corollary 2. *(See Lemma 8.2 in [23, p. 72].) Let $h \in C(R \times R, R)$, $x(t)$ and $y(t)$ be continuous. Furthermore, $x(t)$ is a solution of*

$$D^+x(t) \leqslant h\big(t, x(t)\big), \quad t \geqslant t_0,$$

$y(t)$ is the maximal solution of

$$\dot{y}(t) = h\big(t, y(t)\big), \quad t \geqslant t_0.$$

Then for all $t \geqslant t_0$,

$$x(t) \leqslant y(t),$$

provided that $x(t_0) \leqslant y(t_0)$.

Let $C^{1,2}(R \times R^n, R)$ denote the family of all nonnegative functions $V(t,x)$ on $R \times R^n$ which are twice continuously differentiable in x and once in t. For each $V(t,x) \in C^{1,2}(R \times R^n, R)$, we define an operator $\mathcal{L}V$, associated with the SFDE (5), from $R \times R^n$ to R by

$$\mathcal{L}V(t,x) = V_t(t,x) + V_x(t,x)B(t,x_t) + \frac{1}{2}\operatorname{trace}\big[\sigma^T(t,x_t)V_{xx}\sigma(t,x_t)\big],$$
$$V_t(t,x) = \frac{\partial V(t,x)}{\partial t}, \qquad V_x(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1}, \dots, \frac{\partial V(t,x)}{\partial x_n}\right),$$
$$V_{xx}(t,x) = \left(\frac{\partial V^2(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Theorem 3. *Let the conditions of Theorem 2 hold. Assume that there are functions $V \in C^{1,2}([t_0-\tau,T)\times R^n, R_+^m)$ and $F \in C([t_0,T)\times R_+^m \times R_+^m, R_+^m)$ such that*

$$\max_{1\leqslant i\leqslant m}\Big\{\lim_{|x|\to\infty}\Big[\inf_{t_0-\tau\leqslant t<T} V_i(t,x)\Big]\Big\} = \infty, \tag{40}$$
$$\mathbb{E}\mathcal{L}V(t,x) \leqslant F\big(t, \mathbb{E}V(t,x), \overline{\mathbb{E}V}(t,x)\big), \quad \forall t\in[t_0,T),\ x\in R^n, \tag{41}$$

where $\mathcal{L}V = (\mathcal{L}V_1,\dots,\mathcal{L}V_m)^T$, $\mathbb{E}V = (\mathbb{E}V_1,\dots,\mathbb{E}V_m)^T$ and $R_+ = [0,\infty)$.

Assume moreover that F is an H_m-function and for arbitrary given initial condition, the solution $u(t)$ of the delay differential equation

$$\dot{u}(t) = F\big(t, u(t), \bar{u}(t)\big) \tag{42}$$

exists on $[t_0-\tau,T)$. Then any solution of (5) *and* (6) *exists also on $[t_0-\tau,T)$.*

Proof. From Theorem 2, the SFDE (5) with the initial condition (6) has a unique solution $x(t) = x(t;t_0,\xi)$ with maximum existing interval $[t_0-\tau,\beta)$. Now, we only need to prove $\beta = T$. If $\beta < T$, by Corollary 1, there exists a measurable subset $S\subset\Omega$ with $P(S)>0$ such that $x(t)$ explodes in $[t_0-\tau,\beta)$ for all $\omega\in S$. For any sufficiently large integer n, we define the stopping times

$$\tau_n(t_0,\xi) = \beta \wedge \inf\big\{t\in[t_0,\beta):\ \big|x(t)\big| \geqslant n\big\},$$

where, as usual, we set $\inf\emptyset = \infty$. Clearly, τ_n's are increasing. So they have the limit $\beta = \lim_{n\to\infty}\tau_n$. By Itô's formula, we get

$$V\big(t\wedge\tau_n, x(t\wedge\tau_n)\big) = V\big(t_0, x(t_0)\big) + \int_{t_0}^{t\wedge\tau_n} \mathcal{L}V\big(s,x(s)\big)\,ds$$
$$+ \int_{t_0}^{t\wedge\tau_n} V_{xx}\big(s,x(s)\big)\cdot\sigma\big(s,x(s)\big)\,d\omega(s), \quad t\in[t_0,\beta). \tag{43}$$

From (41) and (43), for small enough $\Delta t>0$, we have

$$\Delta\mathbb{E}V(t\wedge\tau_n) \triangleq \mathbb{E}V\big((t+\Delta t)\wedge\tau_n, x\big((t+\Delta t)\wedge\tau_n\big)\big) - \mathbb{E}V\big(t\wedge\tau_n, x(t\wedge\tau_n)\big)$$

$$= \mathbb{E} \int_{t\wedge\tau_n}^{(t+\Delta t)\wedge\tau_n} \mathcal{L}V\big(s, x(s)\big)\, ds$$

$$\leqslant \int_{t\wedge\tau_n}^{(t+\Delta t)\wedge\tau_n} F\big(s, \mathbb{E}V\big(s, x(s)\big), \overline{\mathbb{E}V}\big(s, x(s)\big)\big)\, ds, \quad t \in [t_0, \beta).$$

Noting $\Delta t > 0$, we have

$$\frac{\Delta \mathbb{E}V(t\wedge\tau_n)}{\Delta t} \leqslant \frac{1}{\Delta t} \int_{t\wedge\tau_n}^{(t+\Delta t)\wedge\tau_n} F\big(s, \mathbb{E}V\big(s, x(s)\big), \overline{\mathbb{E}V}\big(s, x(s)\big)\big)\, ds, \quad t \in [t_0, \beta).$$

Letting $\Delta t \to 0^+$, we get

$$D^+\mathbb{E}V\big(t\wedge\tau_n, x(t\wedge\tau_n)\big) \leqslant F\big(t\wedge\tau_n, \mathbb{E}V\big(t\wedge\tau_n, x(t\wedge\tau_n)\big), \overline{\mathbb{E}V}\big(t\wedge\tau_n, x(t\wedge\tau_n)\big)\big), \tag{44}$$

for all $t \in [t_0, \beta)$. Since F is an H_m-function, by Lemma 2, we obtain that

$$\mathbb{E}V\big(t\wedge\tau_n, x(t\wedge\tau_n)\big) \leqslant u(t\wedge\tau_n), \quad \forall t \in [t_0, \beta),\ \text{for each } n \gg 1,$$

provided that one chooses a suitable initial condition of (42) such that $\mathbb{E}V(t_0 + s, \xi(s)) \leqslant u(t_0 + s)$, for all $s \in [-\tau, 0]$. So,

$$\mathbb{E}V\big(\beta\wedge\tau_n, x(\beta\wedge\tau_n)\big) \leqslant u(\beta\wedge\tau_n), \quad \text{for each } n \gg 1,$$

yielding

$$\mathbb{E} \inf_{t_0-\tau\leqslant t<T} V\big(t, x(\beta\wedge\tau_n)\big) \leqslant u(\beta\wedge\tau_n), \quad \text{for each } n \gg 1. \tag{45}$$

Letting $n \to \infty$, $u(\beta\wedge\tau_n) \to u(\beta)$, and by condition (40),

$$\inf_{t_0-\tau\leqslant t<T} V_{i_0}\big(t, x(\beta\wedge\tau_n)\big) \to \infty, \quad \text{for some } i_0 \in \{1, \dots, m\},\ \forall \omega \in S.$$

From $P(S) > 0$, we have

$$\mathbb{E} \inf_{t_0-\tau\leqslant t<T} V_{i_0}\big(t, x(\beta\wedge\tau_n)\big) \to \infty, \quad \text{for some } i_0 \in \{1, \dots, m\},\ \text{when } n \to \infty.$$

This together with (45) implies that $\infty \leqslant u_{i_0}(\beta)$. Since the solution of (42) exists in $[t_0 - \tau, T)$, this is a contradiction. Consequently, the proof is completed. □

Lemma 3. *If $x(t) \in C([t_0 - \tau, T), R)$, then $V(t) = \max_{-\tau\leqslant s\leqslant 0} x(t+s)$ is a continuous function of t for $t \in [t_0, T)$.*

Proof. For arbitrary $t, t+h \in (t_0, T)$, there are $s_1, s_2 \in [-\tau, 0]$ such that

$$V(t) = \max_{-\tau \leqslant s \leqslant 0} x(t+s) = x(t+s_1), \qquad V(t+h) = \max_{-\tau \leqslant s \leqslant 0} x(t+h+s) = x(t+h+s_2).$$

Then, $x(t+s_1) \geqslant x(t+s_2)$, $x(t+h+s_2) \geqslant x(t+h+s_1)$ and

$$\begin{aligned} \left|V(t) - V(t+h)\right| &= \left|x(t+s_1) - x(t+h+s_2)\right| \\ &\leqslant \begin{cases} x(t+s_1) - x(t+h+s_1), & \text{for } V(t) \geqslant V(t+h), \\ x(t+h+s_2) - x(t+s_2), & \text{for } V(t) \leqslant V(t+h). \end{cases} \end{aligned} \tag{46}$$

Since $x(t)$ is continuous in $[t_0 - \tau, T)$, the right side of (46) approaches to the zero as h tends to the zero. Thus, we have

$$\lim_{h \to 0} V(t+h) = V(t). \tag{47}$$

Consequently, the proof is completed. □

Lemma 4. *Let $U(t, y)$ be continuous and nonnegative for $t_0 \leqslant t < t_0 + a$, $y \geqslant 0$, and $u(t) \in C([t_0 - \tau, t_0 + a), R)$ satisfy*

$$D^+ u(t) \leqslant U(t, \bar{u}), \quad t \in [t_0, t + a). \tag{48}$$

Then

$$D^+ V(t) \leqslant U\big(t, V(t)\big), \quad t \in [t_0, t + a),$$

where $V(t) = \bar{u} = \max_{-\tau \leqslant s \leqslant 0} u(t+s)$.

Proof. From $u(t) \in C([t_0 - \tau, t_0 + a), R)$ and Lemma 3, $V(t) = \bar{u}$ is continuous on $[t_0, t_0 + a)$. Then $D^+ V(t)$ exists. For each given $t \in [t_0, t_0 + a)$, we denote

$$t^* = \sup\big\{\hat{t} \in [t - \tau, t] \,\big|\, V(t) = u(\hat{t})\big\}.$$

By the continuity of $u(t)$, we have $V(t) = u(t^*)$.

(1) If $t^* < t$, then from the definitions of $V(t)$ and t^*, we obtain

$$u(\tilde{t}) < u\big(t^*\big), \quad \forall \tilde{t} \in \big(t^*, t\big], \quad \text{and} \quad u(\tilde{t}) \leqslant u\big(t^*\big), \quad \forall \tilde{t} \in \big[t - \tau, t^*\big].$$

Since $u(t) \in C([t_0 - \tau, t_0 + a), R)$, we can obtain that for small enough $h > 0$,

$$u(\tilde{t}) \leqslant u\big(t^*\big), \quad \forall \tilde{t} \in [t - \tau, t + h].$$

So, $V(t+h) \leqslant u(t^*) = V(t)$. This implies that

$$D^+ V(t) = \limsup_{h \to 0^+} \frac{V(t+h) - V(t)}{h} \leqslant 0.$$

(2) If $t^* = t$, then $V(t) = u(t)$. So, for small enough $h > 0$, there is a $\bar{t} \in [t, t+h]$ such that $V(t+h) = u(\bar{t})$, $\bar{t} \to t$ as $h \to 0^+$ and $u(\bar{t}) \geqslant u(t)$. Therefore, by (48),

$$\begin{aligned} D^+V(t) &= \limsup_{h \to 0^+} \frac{u(\bar{t}) - u(t)}{h} \leqslant \limsup_{\bar{t} \to t^+} \frac{u(\bar{t}) - u(t)}{\bar{t} - t} \\ &= D^+u(t) \leqslant U(t, \bar{u}) = U\big(t, V(t)\big). \end{aligned}$$

So, we have $D^+V(t) \leqslant U(t, V(t))$. This completes the proof. □

Theorem 4. *Let the conditions of Theorem* 2 *hold. Assume that there are functions* $V \in C^{1,2}([t_0 - \tau, T) \times R^n, R_+)$ *and* $F \in C([t_0, T) \times R_+, R_+)$ *such that*

$$\begin{gathered} \lim_{|x| \to \infty} \Big[\inf_{t_0 - \tau \leqslant t < T} V(t, x) \Big] = \infty, \\ \mathbb{E}\mathcal{L}V(t, x) \leqslant F\big(t, \overline{\mathbb{E}V}(t, x)\big), \quad \forall t \in [t_0, T),\ x \in R^n. \end{gathered} \tag{49}$$

Assume moreover that for arbitrary given initial condition, the maximal solution $u(t)$ *of the differential equation*

$$\dot{u}(t) = F\big(t, u(t)\big) \tag{50}$$

exists on $[t_0, T)$. *Then any solution of* (5) *and* (6) *exists also on* $[t_0 - \tau, T)$.

Proof. Using the same method in (44), we obtain

$$D^+\mathbb{E}V\big(t \wedge \tau_n, x(t \wedge \tau_n)\big) \leqslant F\big(t \wedge \tau_n, \overline{\mathbb{E}V}\big(t \wedge \tau_n, x(t \wedge \tau_n)\big)\big), \quad t \in [t_0, \beta).$$

By Lemma 4, we obtain

$$D^+v(t) \leqslant F\big(t, v(t)\big), \quad t \in [t_0, \beta), \tag{51}$$

where $v(t) = \overline{\mathbb{E}V}(t \wedge \tau_n, x(t \wedge \tau_n))$. From (50), (51) and Corollary 2, we have

$$v(t) \leqslant u(t), \quad t \in [t_0, \beta),$$

provided that $v(t_0) \leqslant u(t_0)$. The remainder of proof is the same with the one in Theorem 3. □

Remark 4. Theorem 4 is a generalization of Wintner theorem in [20, p. 29] or [26]. In the proof of Theorems 3 and 4, the inequalities (41) and (49) were used under considering an explosion of the solutions $x(t)$. Then, Theorems 3 and 4 are still valid if the inequalities (41) and (49) hold for all $x \in R^n$ with $|x| \geqslant r$ for some positive constant r. Therefore, we can get the following corollary.

Corollary 3. *Assume that all conditions of Theorem* 4 *are satisfied except that the inequality* (49) *is replaced by the following inequality*

$$\mathcal{L}V(t,x) \leqslant F\big(t, \overline{V}(t,x)\big), \tag{52}$$

for all $t \in [t_0, T)$, $x \in R^n$ with $|x| \geqslant r$, where r is a positive constant. Moreover, assume that $F(t,u) \in C([t_0,T) \times R_+, R_+)$ is concave with respect to $u \in R_+$ for each fixed $t \in [t_0,T)$. Then the same conclusion of Theorem 4 *holds.*

Proof. Since $F(t,u)$ is concave with respect to u, the inequality (52) implies that the inequality (49) holds. So, the conclusion of Corollary 3 is true by Theorem 4 and Remark 4. □

Remark 5. When $\tau = 0$, the SFDE (5) becomes a SODE. For SODEs, the quasi-boundedness in Theorems 2–4 and Corollary 3 may be taken out since the boundedness of a deterministic continuous function on a closed bounded region in Euclidean space is of course satisfied. Therefore, Corollary 3 is natural generalization of Theorem 1 in [7] when $\tau = 0$ and $t_0 = 0$.

Remark 6. In Theorem 4 and Corollary 3, the assumption that the maximal solution of (50) exists on $[t_0, T)$ is easy to reach. The following lemma will be an example.

Lemma 5. *Let $F(t,u) = a(t) + b(t)k(u)$ in $[t_0,T) \times R_+$, where $a(t), b(t) \in C([t_0,T), R_+)$ and $k(u) \in C(R_+, R_+)$ satisfying*

$$\int_0^{+\infty} \frac{du}{1+k(u)} = +\infty. \tag{53}$$

Then the maximal solution of (50) *exists on* $[t_0, T)$.

Proof. By the continuousness of $F(t,u)$, the maximal solution denoted by $u(t)$ of (50) exists. Assume the maximum existing interval of $u(t)$ is $[t_0, \delta)$ and $\delta < T$, then

$$u(t) \to +\infty, \quad \text{when } t \to \delta^-. \tag{54}$$

Since $a(t), b(t) \in C([t_0,T), R_+)$ and $\delta < T$, there exists a constant $M > 0$ such that

$$a(t) \leqslant M, \qquad b(t) \leqslant M, \quad \forall t \in [t_0, \delta].$$

So, we obtain

$$\dot{u}(t) = F\big(t, u(t)\big) = a(t) + b(t)k\big(u(t)\big) \leqslant M\big(1 + k\big(u(t)\big)\big), \quad \forall t \in [t_0, \delta),$$

yielding

$$\frac{\dot{u}(t)}{1+k(u(t))} \leqslant M, \quad \forall t \in [t_0, \delta).$$

Thus,

$$\int_{t_0}^{t} \frac{\dot{u}(t)}{1+k(u(t))}\, ds \leqslant M(t-t_0), \quad \forall t \in [t_0, \delta),$$

yielding

$$\int_{u(t_0)}^{u(t)} \frac{du}{1+k(u)} \leqslant M(t-t_0), \quad \forall t \in [t_0, \delta), \tag{55}$$

where $u(t_0) \geqslant 0$. Let $t \to \delta^-$ in (55). Then, from (53)–(55), we obtain $+\infty \leqslant M(\delta - t_0)$. This is a contradiction. So, $\delta = T$. Consequently, the proof is completed. □

Corollary 4. *Suppose that the conditions of Theorem* 2 *hold and there exist functions* $M(t), N(t) \in C([t_0, T), R_+)$ *such that*

$$\mathcal{L}\big|x(t)\big| \leqslant M(t) + N(t)\|x_t\|, \quad \forall t \in [t_0, T),\ x \in R^n. \tag{56}$$

Then any solution of (5) *and* (6) *exists also on* $[t_0 - \tau, T)$.

Proof. Obviously, the linear function $F(t,u) = M(t) + N(t)u$ is concave with respect to $u \in R_+$ for each fixed $t \in [t_0, T)$. Thus, letting $V(t,x) = |x|$, the conclusion of Corollary 4 can be implied by Corollary 3 and Lemma 5. □

In order to compare with the known results, we introduce the following lemma.

Lemma 6. *If there is the right-hand derivative* $\dot{x}(t)$ *at t for a continuous function* $x(t)$*, we have*

$$D^+\big|x(t)\big| \leqslant \big|\dot{x}(t)\big|.$$

Proof. From the definition of right-hand derivative, for $h > 0$ small enough

$$\dot{x}(t) = \big(x(t+h) - x(t)\big)/h + o(h).$$

That is,

$$x(t+h) = h\dot{x}(t) + x(t) + o(h)h.$$

Thus

$$\begin{aligned} D^+\big|x(t)\big| &= \limsup_{h\to 0^+}\big[\big|\dot{x}(t)h + x(t) + o(h)h\big| - \big|x(t)\big|\big]/h \\ &\leqslant \limsup_{h\to 0^+}\big[\big|\dot{x}(t)h + x(t) + o(h)h - x(t)\big|\big]/h = \big|\dot{x}(t)\big|. \qquad \square \end{aligned}$$

Remark 7. All theorems and corollaries in this section are still valid for the FDE (26) if the conditions on stochastic variable are taken out (including the mathematical expectation), the differentiable conditions on V are weakened as $V \in C([t_0 - \tau, T) \times R^n, R_+^m)$ (or $V \in C([t_0 - \tau, T) \times R^n, R_+))$ and the operator $\mathcal{L}$ is replaced by the upper right derivative D^+ (in fact, for the FDE (26), we can get (44) by direct calculating $D^+ V$). In this way, Corollary 4 is a generalization of Corollary D in [21, p. 308]. In fact, for the FDE (26), the right-hand derivative of $x(t)$ exists [22, p. 38] and the inequality (56) becomes

$$D^+ \left| x(t) \right| \leqslant M(t) + N(t) \|x_t\|, \quad \forall t \in [t_0, T), \ x \in R^n. \tag{57}$$

From Lemma 6 and the FDE (26), we have

$$D^+ \left| x(t) \right| \leqslant \left| \dot{x}(t) \right| = \left| f(t, x_t) \right|.$$

Thus, the inequality (57) holds if the condition (5) in [21, p. 308] holds, that is,

$$\left| f(t, x_t) \right| \leqslant M(t) + N(t) \|x_t\|, \quad \forall t \in [t_0, T), \ x \in R^n.$$

Therefore, all conditions of Corollary 4 for the FDE (26) can be implied by Corollary D in [21, p. 308].

Remark 8. The methods in this paper can be applied to the stochastic differential equations with infinite delays and stochastic partial functional differential equations. They will appear in our next publications.

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